Name of College - ...
J. Road
Deptt - Mathematics

Topic - Indefinite Integrals (Problem) Contd.
Time - 10.15 A.M To 11:00 A.M
11:00 A.M To 11:45 A.M
Class - B.Sc I (Hons)

By - Dr. Shree Nath Sharma

Important Integral
Integrals

(i) $\int \frac{1-x^2}{1+x^2\sqrt{1+x^4}}\,dx$

(ii) $\int \frac{1+x^2}{1-x^2\sqrt{1+x^4}}\,dx$

(iii) $\int \frac{\sqrt{1+x^4}}{1-x^4}\,dx$

(iv) $\int \frac{x^2\,dx}{\sqrt{1+x^4}\,(1-x^4)}$

(v) $\int \frac{x-1}{(x+1)\sqrt{x(x^2+x+1)}}\,dx$

(vi) $\int \frac{1+x^2}{1-x^2}\cdot\frac{dx}{\sqrt{1+x^2+x^4}}$

(vii) $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}$

(1) $\int \frac{1-x^2}{1+x^2\sqrt{1+x^4}}\,dx$

$$= \int \frac{x\left(\frac{1}{x}-x\right)}{x\left(\frac{1}{x}+x\right)\sqrt{x^2\left(\frac{1}{x^2}+x^2\right)}}\,dx$$

PAGE NO. 2
DATE 02-09-2021

$$= \int \frac{x\left(\frac{1}{x}-x\right)}{x^2\left(\frac{1}{x}+x\right)\sqrt{\frac{1}{x^2}+x^2}}\,dx \qquad \int \frac{\left[\frac{1}{x}-x\right]}{x\left[\frac{1}{x}+x\right]\sqrt{\frac{1}{x^2}+x^2}}\,dx$$

Put $x+\frac{1}{x}=t \Rightarrow \left[1-\frac{1}{x^2}\right]dx = dt$

$$\Rightarrow \left(x-\frac{1}{x}\right)\frac{dx}{x} = dt$$

$$\Rightarrow \left(\frac{1}{x}-x\right)\frac{dx}{x} = -dt$$

Also $x^2+\frac{1}{x^2}+2 = t^2$

$$\Rightarrow x^2+\frac{1}{x^2} = t^2-2$$

$$\therefore = \int \frac{-dt}{t\sqrt{t^2-2}} \qquad \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x = -\operatorname{cosec}^{-1}x$$

$$= \frac{1}{\sqrt{2}}\operatorname{cosec}^{-1}\frac{t}{\sqrt{2}}+C \qquad \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}}{t}+C \qquad \operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}}{x+\frac{1}{x}}+C$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}\,x}{x^2+1}+C$$

(11) Integrate $\int \frac{1+x^2}{1-x^2\sqrt{1+x^4}}\,dx$

$$I = \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}}\,dx$$

$$= \int \frac{x\left(x+\frac{1}{x}\right)}{x\left(\frac{1}{x}-x\right)\sqrt{x^2\left(\frac{1}{x^2}+x^2\right)}}\,dx$$

$$= \int \frac{\left(x+\frac{1}{x}\right)}{x\left(\frac{1}{x}-x\right)\sqrt{\frac{1}{x^2}+x^2}}\,dx$$

PAGE NO : 3
DATE : 02-09-2020

(111)

$$= -\int \frac{\left(x+\frac{1}{x}\right)}{x\left(x-\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}}\,dx$$

Put $x - \frac{1}{x} = z \Rightarrow \left[1+\frac{1}{x^2}\right] dx = dz$

$$\frac{x\left(1+\frac{1}{x^2}\right)}{x}\,dx = dz$$

$$\frac{\left(x+\frac{1}{x}\right)dx}{x} = dz$$

$$x^2 + \frac{1}{x^2} - 2 = \left(x-\frac{1}{x}\right)^2$$

$$x^2+\frac{1}{x^2}-2 = z^2 \Rightarrow x^2+\frac{1}{x^2} = z^2+2$$

$$\therefore I = -\int \frac{dz}{z\sqrt{z^2+2}}$$

Again Put $z = \frac{1}{t}$

$$z^2+2 = \frac{1}{t^2}+2$$

$$= 2t^2+1$$

$$dz = -\frac{1}{t^2}\,dt$$

$$= -\int \frac{t\,dt}{t^2 \cdot \sqrt{\frac{1}{t^2}+2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2+\frac{1}{2}}}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left[x+\sqrt{x^2+a^2}\right]$$

$$= \frac{1}{\sqrt{2}}\log\left[t+\sqrt{t^2+\frac{1}{2}}\right]+C$$

$$= \frac{1}{\sqrt{2}}\log\left[\frac{1}{z}+\sqrt{\frac{1}{z^2}+\frac{1}{2}}\right]+C$$

$$= \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+\sqrt{2+z^2}}{z\sqrt{2}}+C$$

$$= \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+\sqrt{x^2+\frac{1}{x^2}}}{\left(x-\frac{1}{x}\right)\sqrt{2}}+C = \frac{1}{\sqrt{2}}\log\frac{x\sqrt{2}+\sqrt{x^4+1}}{(x^2-1)\sqrt{2}}+C$$

(III) Integrate : $\int \frac{\sqrt{1+x^4}}{1-x^4}\,dx$

$$= \int \frac{(1+x^4)}{(1-x^4)\sqrt{1+x^4}}\,dx$$

Combination of problem (I) and (II)

$$= \frac{1}{2}\int \left\{ \frac{1-x^2}{1+x^2} + \frac{1+x^2}{1-x^2} \right\} \frac{dx}{\sqrt{1+x^4}}$$

$$= \frac{1}{2}\int \frac{1-x^2}{1+x^2}\cdot\frac{dx}{\sqrt{1+x^4}} + \frac{1}{2}\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}}\,dx$$

using Example (I) and (II)

$$= \frac{1}{2\sqrt{2}} \sin^{-1}\frac{\sqrt{2}\,x}{x^2+1} + \frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}\,x + \sqrt{x^4+1}}{\sqrt{2}\,(x^2-1)} + C$$

(IV) $I = \int \frac{x^2\,dx}{(1-x^4)\sqrt{1+x^4}}$

We have the given integral

$$= \frac{1}{4}\int \left[\frac{1+x^2}{1-x^2} - \frac{1-x^2}{1+x^2} \right] \frac{dx}{\sqrt{1+x^4}}$$

$$= \frac{1}{4}\int \frac{1+x^2}{1-x^2\sqrt{1+x^4}}\,dx - \frac{1}{4}\int \frac{1-x^2}{(1+x^2)}\cdot\frac{dx}{\sqrt{1+x^4}}$$

$$= \frac{1}{4\sqrt{2}} \log \frac{\sqrt{2}\,x + \sqrt{x^4+1}}{\sqrt{2}\,(x^2-1)} - \frac{1}{4\sqrt{2}} \sin^{-1}\frac{\sqrt{2}\,x}{x^2+1} + C$$

using Problem (I) and (II)

(V) $\int \frac{x-1}{x+1}\,\frac{1}{\sqrt{x(x^2+x+1)}}\,dx$

We have, the given integral

$$= \int \frac{x-1}{x+1} \frac{dx}{x\sqrt{x+1+\frac{1}{x}}}$$

$$= \int \frac{(x^2-1)}{(x+1)^2} \frac{dx}{x\sqrt{x+1+\frac{1}{x}}}$$

$$= \int \frac{x}{(x+1)^2\sqrt{x+1+\frac{1}{x}}} \cdot \frac{x^2-1}{x^2}\, dx$$

Put $x+1+\frac{1}{x} = z^2$

$$\left(1-\frac{1}{x^2}\right) dx = 2z\, dz$$

$$z^2+1 = x+1+\frac{1}{x}+1 = \frac{x^2+2x+1}{x} = \frac{(x+1)^2}{x}$$

∴ the given integral $= \int \frac{2z\, dz}{(z^2+1)\, z}$

$$= 2\int \frac{dz}{z^2+1}$$

$$= 2\tan^{-1} z + C$$

$$= 2\tan^{-1}\sqrt{\frac{x^2+x+1}{x}} + C$$

x (VI) $\int \frac{1+x^2}{1-x^2\sqrt{1+x^2+x^4}}\, dx$

we have

$$I = \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^2+x^4}}\, dx$$

PAGE NO 6
DATE 02-09-2020

$$= \int \frac{x\left(x+\frac{1}{x}\right)dx}{x\left(\frac{1}{x}-x\right)\cdot x\sqrt{\frac{1}{x^2}+1+x^2}}$$

Put $x - \frac{1}{x} = z$

$$\left(1+\frac{1}{x^2}\right)dx = dz$$

$$\text{or } \left(x+\frac{1}{x}\right)\frac{dx}{x} = dz$$

$$x^2 + \frac{1}{x^2} - 2 = z^2$$

$$x^2 + \frac{1}{x^2} = z^2 + 2$$

$$x^2 + \frac{1}{x^2} + 1 = z^2 + 3$$

The Given integral

$$I = -\int \frac{dz}{z\sqrt{z^2+3}}$$

Put $z = \frac{1}{y}$

$$dz = -\frac{1}{y^2}dy$$

$$\therefore I = -\int \frac{\frac{1}{y^2}dy}{-\frac{1}{y}\sqrt{\frac{1}{y^2}+3}} = \int \frac{dy}{\sqrt{1+3y^2}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{dy}{\sqrt{y^2+\frac{1}{3}}}$$

$$= \frac{1}{\sqrt{3}} \sinh^{-1}\frac{y}{1/\sqrt{3}} + C = \frac{1}{\sqrt{3}} \sinh^{-1}\left(\sqrt{3}\cdot\frac{1}{z}\right) + C$$

$$= \frac{1}{\sqrt{3}} \sinh^{-1} \frac{\sqrt{3}x}{x^2-1} + C$$

(VII) Integrate

$$\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx$$

The given integral $= \int \frac{x^2\left(x^2-\frac{1}{x^2}\right)}{x^2\sqrt{x^4+x^2+1}}\,dx$

$$= \int \frac{x^2-\frac{1}{x^2}}{x\sqrt{x^2+1+\frac{1}{x^2}}}\,dx$$

Put $x^2+\frac{1}{x^2} = z$

$$\left(2x - 2\frac{1}{x^3}\right)dx = dz$$

or $2\left[x-\frac{1}{x^3}\right]dx = dz$

or $\frac{2}{x}\left(x^2-\frac{1}{x^2}\right)dx = dz$

$\Rightarrow \frac{\left(x^2-\frac{1}{x^2}\right)}{x}\,dx = \frac{1}{2}\,dz$

$$= \frac{1}{2}\int \frac{dz}{\sqrt{z+1}}$$

$$= \frac{1}{2}\int (z+1)^{-1/2}\,dz$$

$$= \frac{1}{2}\,\frac{(z+1)^{1/2}}{1/2} + C$$

$$= \sqrt{z+1} + C$$

$$= \sqrt{x^2+\frac{1}{x^2}+1} + C$$

$$= \frac{\sqrt{x^4+x^2+1}}{x} + C$$